

**A KRASNOSELSKII-TYPE ALGORITHM FOR
APPROXIMATING A COMMON FIXED POINT OF
A FINITE FAMILY OF MULTIVALUED STRICTLY
PSEUDO CONTRACTIVE MAPPINGS
IN HILBERT SPACES**

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Abstract

Let K be a nonempty closed and convex subset of a real Hilbert space H and let $T_1, \dots, T_m : K \rightarrow 2^K$ be m multivalued strictly pseudo contractive mappings. A Krasnoselskii-type algorithm is constructed and the corresponding sequence $\{x_n\}$ is proved to be an approximating fixed point sequence of each T_i , i.e., $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then, convergence theorems are also proved under appropriate additional conditions. Finally, application to optimization problem with constraints is given.

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1. Introduction

Let (X, d) be a metric space, K be a nonempty subset of X , and $T : K \rightarrow 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well-known mathematicians (see, for example, Brouwer [2]; Kakutani [3]; Nash [7, 8]; Geanakoplos [17]; Nadla [20]; Downing and Kirk [4]).

Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in *game theory and market economy* and in other areas of mathematics, such as in *non-smooth differential equations and differential inclusions, optimization theory*. We describe briefly the connection of fixed point theory for *multi-valued* mappings with these applications.

1.1. Optimization problems with constraints

Let H be a real Hilbert space H and $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\varphi : H \rightarrow 2^H$ be a multivalued mapping. Consider the following optimization problem:

$$(P) \quad \begin{cases} \min f(x) \\ 0 \in \varphi(x). \end{cases}$$

It is known that the multivalued map, ∂f , the subdifferential of f , is *maximal monotone* (see, e.g., [6]), where for $x, w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle \quad \forall y \in H \\ &\Leftrightarrow x \in \arg \min(f - \langle \cdot, w \rangle). \end{aligned}$$

Set $T_2 := \varphi$. It is easily seen that, for $x \in H$, with $0 \in \varphi(x)$, x is a solution of (P) if and only if $0 \in \partial f(x) \cap \varphi(x)$, or equivalently,

$$x \in T_1 x \cap T_2 x,$$

with $T_1 := I - \partial f$ and $T_2 := I - \varphi$, where I is the identity map of H . Therefore, x is a solution of (P) if and only if x is a common fixed point of the multivalued mappings T_1 and T_2 .

1.2. Game theory and market economy

In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [7, 8] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [2] or Kakutani [3] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued map*, whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, e.g., [7]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n . $n = 1, \dots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the n -th player and is assumed to be nonempty, compact, and convex and $u_n : K := K_1 \times K_2 \times \dots \times K_N \rightarrow \mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$. For each n , $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \dots \times K_{n-1} \times K_{n+1} \times \dots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \dots, \sigma_{n-1}, \sigma_{n+1}, \dots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \dots, \sigma_{n-1}, z_n, \sigma_{n+1}, \dots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n -th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} , if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \rightarrow 2^{K_n}$ be the multivalued map defined by

$$T_n(\sigma_{-n}) := \text{Arg max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \quad \forall \sigma_{-n} \in K_{-n}.$$

Definition. A collective action $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n , $\bar{\sigma}_n$ is the best response for the n -th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n ,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}), \quad (1.1)$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \quad (1.2)$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the multivalued map $T : K \rightarrow 2^K$ defined by

$$T(\sigma) := [T_1(\sigma_{-1}), T_2(\sigma_{-2}), \dots, T_N(\sigma_{-N})].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of *fixed point theory of multi-valued mappings*. However, it has been remarked that the applications of this theory to equilibrium are mostly static: They enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. *This is part of the problem that is being addressed by iterative methods for fixed point of multi-valued mappings.*

1.3. Differential inclusions ([24]). For $\Omega = (0, \pi)$, consider the following differential inclusion:

$$\begin{cases} -\frac{d^2u}{dt^2} \in u - \frac{1}{4} - \frac{1}{4} \operatorname{sgn}(u - 1), & t \in \Omega; \\ u(0) = 0; \\ u(\pi) = 0, \end{cases} \quad (1.3)$$

where

$$\operatorname{sgn}(x) := \begin{cases} -1, & \text{if } x < 0; \\ [-1, 1], & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

Let $H := H_0^1(\Omega)$ and $(\cdot, \cdot)_H$ the inner product on H defined by

$$(u, v)_H = \int_{\Omega} u'v' dt, \quad \forall u, v \in H.$$

From Riesz theorem, there exists an operator $A : H \rightarrow H$ satisfying

$$(Au, v)_H = \int_{\Omega} u'v' dt, \quad \forall v \in H.$$

For $u \in H$, let $E(u) := u - \frac{1}{4} - \frac{1}{4} \operatorname{sgn}(u - 1)$. For $w \in E(u)$, let

$L_u^w : H \rightarrow \mathbb{R}$ the map defined by

$$L_u^w(v) := \int_{\Omega} wv dt, \quad \forall v \in H.$$

Then L_u^w is linear and continuous on H . Therefore, using again Riesz theorem, there exists a unique vector $b_u^w \in H$ such that

$$(b_u^w, v)_H = \int_{\Omega} wv dt, \quad \forall v \in H.$$

Let $B : H \rightarrow 2^H$ be the multivalued map defined by

$$Bu = \{b_u^w : w \in E(u)\}.$$

Then, u is a solution of (1.3) if and only if $Au \in Bu$. Further, the operator $A : H \rightarrow H$ is strongly monotone. If we introduce the multivalued map $T : H \rightarrow 2^H$ defined by

$$Tu = u - Au - Bu, \quad \forall u \in H.$$

Then $u \in H$ is a solution of (1.3) if and only if $u \in Tu$, that is, u is a fixed point of T .

Let K be a nonempty subset of a normal space E . The set K is called *proximal* (see, e.g., [29, 28, 31]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in K\} = d(x, K),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets and nonempty, proximal and bounded subsets of K , respectively. The *Hausdorff metric* on $CB(K)$ is defined by

$$D(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},$$

for all $A, B \in CB(K)$. A multi-valued mapping $T : D(T) \subseteq E \rightarrow CB(E)$ is called *L-Lipschitzian*, if there exists $L > 0$ such that

$$D(Tx, Ty) \leq L\|x - y\| \quad \forall x, y \in D(T). \quad (1.4)$$

When $L \in (0, 1)$ in (1.4), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Several papers deal with the problem of *approximating* fixed points of *multi-valued nonexpansive* mappings (see, e.g., [33, 32, 29, 28, 31], and the references therein) and their generalizations (see, e.g., [34]).

The important class of single-valued *k*-strictly pseudo-contractive maps on Hilbert spaces was introduced by Browder and Petryshyn [1] as a generalization of the class of nonexpansive mappings.

Definition 1.1. Let K be a nonempty subset of a real Hilbert space H . A map $T : K \rightarrow H$ is called *k*-strictly pseudo-contractive, if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \quad \forall x, y \in K. \quad (1.5)$$

Motivated by approximating fixed points of multivalued mappings, Chidume et al. [26] introduced the following important class of *multi-valued strictly pseudo-contractive mappings in real Hilbert spaces*, which is more general than the class of *multi-valued nonexpansive mappings*.

Definition 1.2. A multi-valued mapping $T : D(T) \subseteq H \rightarrow CB(H)$ is said to be *k*-strictly pseudo-contractive, if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have

$$(D(Tx, Ty))^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \quad \forall u \in Tx, v \in Ty. \quad (1.6)$$

If $k = 1$ in (1.6), the map T is said to be *pseudo-contractive*.

Remark 1. It is easily seen that any multivalued nonexpansive mapping is *k*-strictly pseudo-contractive for any $k \in (0, 1)$. Moreover, the inverse is not true (see, e.g., Djitte et al. [12]).

Then, they proved strong convergence theorems for this class of mappings. The recursion formula used in [26] is of the *Krasnoselskii-type* [36], which is known to be superior (see, e.g., Remark 4 in [26]) to the recursion formula of Mann [13] or Ishikawa [14]. In fact, they proved the following theorem:

Theorem CA1 (Chidume et al. [26]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H . Suppose that $T : K \rightarrow CB(K)$ is a multi-valued *k*-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$.*

Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$, $x_{n+1} = (1 - \lambda)x_n + \lambda y_n$, $n \geq 0$, where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

They also proved in [26], under some mild compactness conditions or on the map T or on the domain K that the sequence $\{x_n\}$ converges strongly to a fixed point of T .

On the other hand, Abbas et al. [33] introduced a new one-step iterative process for approximating a common fixed point of two multivalued *nonexpansive mappings* in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let $S, T : KCB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \end{cases} \quad (1.7)$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ are such that

$$\begin{cases} \|y_n - p\| \leq d(p, Sx_n); \\ \|z_n - p\| \leq d(p, Tx_n), \end{cases} \quad (1.8)$$

and $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$.

Recently, following the work of Abbas et al. [33], Rashwan and Altwqi [21] introduced a new scheme for approximation a common fixed point of three multivalued nonexpansive mappings in uniformly convex Banach spaces. Let $T, S, R : K \rightarrow CB(K)$ be three multivalued nonexpansive mappings. They employed the following iterative process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n y_n + b_n z_n + c_n w_n, \quad n \geq 1, \end{cases} \quad (1.9)$$

where $y_n \in Tx_n$, $z_n \in Sx_n$, and $w_n \in Rx_n$ are such that

$$\begin{cases} \|y_n - y_{n+1}\| \leq D(Tx_n, Tx_{n+1}) + \eta_n; \\ \|z_n - z_{n+1}\| \leq D(Sx_n, Sx_{n+1}) + \eta_n; \\ \|w_n - w_{n+1}\| \leq D(Rx_n, Rx_{n+1}) + \eta_n, \end{cases} \quad (1.10)$$

and $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$.

Before we state the result of Rashwan and Altwqi [21], we need the following definition:

The mappings $T, S, R : K \rightarrow CB(K)$ are said to satisfy condition (C) if $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Sx$ or $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Rx$, or $d(x, y) \leq d(z, y)$, for $z \in Rx$ and $y \in Sx$.

Let $F = F(T) \cap F(S) \cap F(R)$ be the set of all common fixed points of the mappings T, S , and R .

Theorem RA (Rashwan and Altwqi [21]). *Let E be a uniformly convex Banach space and K be a nonempty closed and convex subset of E . Let $T, S, R : KCB(K)$ be multivalued nonexpansive mappings satisfying condition (C) and $\{x_n\}$ be the sequence defined (1.9) and (1.10). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then*

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 = \lim_{n \rightarrow \infty} d(x_n, Rx_n).$$

It is our purpose in this paper to introduce a new iteration process and prove strong convergence theorems for approximating a common fixed point of a finite family of multivalued strictly pseudo contractive mappings in a real Hilbert space. The iteration process using here is simpler than the one given by (1.7) and (1.8) and the one given by (1.9) and (1.10). The class of mappings using in our theorems is much more larger than that of multivalued nonexpansive mappings. Our theorems generalize and extend those of Abbas et al. [33]; Rashwan and Altwqi [21]; Chidume et al. and many other important results.

2. Preliminaries

In the sequel, we shall need the following definitions and results:

Definition 2.1. Let E be a normal linear space and $T : D(T) \subset E \rightarrow 2^E$ be a multi-valued mapping. The multi-valued mapping $(I - T)$ is said to be *strongly demiclosed* at 0 (see, e.g., [34]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges strongly to x^* and $d(x_n, Tx_n)$ converges to 0, then $d(x^*, Tx^*) = 0$.

Definition 2.2. A mapping $T : K \rightarrow CB(K)$ is said to satisfy condition (I), if there exists a strictly increasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T))), \quad \forall x \in D(T).$$

Lemma 2.3 ([26]). *Let E be a reflexive real Banach space and let $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A, B). \quad (2.1)$$

Proposition 2.4 ([26]). *Let K be a nonempty subset of a real Hilbert space H and let $T : K \rightarrow CB(K)$ be a multi-valued k -strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then T is Lipschitzian.*

Lemma 2.5 ([26]). *Let K be a nonempty closed subset of a real Hilbert space E and let $T : K \rightarrow P(K)$ be a k -strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then $(I - T)$ is strongly demiclosed at zero.*

3. Main Results

Let $m \geq 1$, K be a nonempty closed convex subset of a real Hilbert space H . Let $T_1, \dots, T_m : K \rightarrow CB(K)$ be multivalued k_i -strictly pseudocontractive mappings. Let $\{x_n\}$ be a sequence defined iteratively as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \lambda_0 x_n + \lambda_1 u_n^1 + \dots + \lambda_m u_n^m, \end{cases} \quad (3.1)$$

where $u_n^i \in T_i x_n$, $i = 1, \dots, m$, $\lambda_i \in (0, 1)$ and $\lambda_0 \in (k, 1)$ with $k := \max\{k_i : i = 1, \dots, m\}$ such that $\lambda_1 + \dots + \lambda_m = 1$. In the sequel, we will write $F := \bigcap_{i=1}^m F(T_i)$ for the set of all common fixed points of the mappings T_i , $i = 1, \dots, m$.

We start with the following important lemma:

Lemma 3.1. *Let H be a real Hilbert space and let $\lambda_i \in (0, 1)$, $i = 1, \dots, n$*

such that $\sum_{i=1}^n \lambda_i = 1$. Then,

$$\left\| \sum_{i=1}^n \lambda_i u_i \right\|^2 = \sum_{i=1}^n \lambda_i \|u_i\|^2 - \sum_{i < j} \lambda_i \lambda_j \|u_i - u_j\|^2, \quad \forall u_1, u_2, \dots, u_n \in H. \quad (3.2)$$

Proof. The proof is by induction. It is easily seen that (3.2) is true for $n = 2$. Now assume that (3.2) is true for $n - 1$, with $n \geq 3$. Using the induction assumption, we have

$$\begin{aligned} \left\| \sum_{i=1}^n \lambda_i u_i \right\|^2 &= (1 - \lambda_n)^2 \left\| \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} u_i \right\|^2 \\ &= (1 - \lambda_n)^2 \left\| \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} u_i + \frac{\lambda_n}{1 - \lambda_n} u_n \right\|^2 \end{aligned}$$

$$\begin{aligned}
&= (1 - \lambda_n)^2 \left(\left\| \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} u_i \right\|^2 + \frac{\lambda_n^2}{(1 - \lambda_n)^2} \|u_n\|^2 \right. \\
&\quad \left. + 2 \sum_{i=1}^{n-1} \frac{\lambda_i \lambda_n}{(1 - \lambda_n)^2} \langle u_i, u_n \rangle \right).
\end{aligned}$$

Using the fact that $2\langle u_i, u_n \rangle = \|u_i\|^2 + \|u_n\|^2 - \|u_i - u_n\|^2$, it then follows that

$$\begin{aligned}
\left\| \sum_{i=1}^n \lambda_i u_i \right\|^2 &= (1 - \lambda_n)^2 \left\| \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} u_i \right\|^2 + \lambda_n^2 \|u_n\|^2 + \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_i\|^2 \\
&\quad + \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_n\|^2 - \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_i - u_n\|^2.
\end{aligned}$$

Observing that $\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_1} = 1$ and using the induction assumption, we have

$$\begin{aligned}
\left\| \sum_{i=1}^n \lambda_i u_i \right\|^2 &= (1 - \lambda_n)^2 \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} \|u_i\|^2 - \sum_{i < j}^{n-1} \lambda_i \lambda_j \|u_i - u_j\|^2 - \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_i - u_n\|^2 \\
&\quad + \lambda_n^2 \|u_n\|^2 + \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_i\|^2 + \sum_{i=1}^{n-1} \lambda_i \lambda_n \|u_n\|^2 \\
&= (1 - \lambda_n) \sum_{i=1}^{n-1} \lambda_i \|u_i\|^2 - \sum_{i < j}^n \lambda_i \lambda_j \|u_i - u_j\|^2 + \lambda_n^2 \|u_n\|^2 \\
&\quad + \lambda_n (1 - \lambda_n) \|u_n\|^2 + \lambda_n \sum_{i=1}^{n-1} \lambda_i \|u_i\|^2 \\
&= \sum_{i=1}^n \lambda_i \|u_i\|^2 - \sum_{i < j}^n \lambda_i \lambda_j \|u_i - u_j\|^2,
\end{aligned}$$

which ends the proof. \square

We now prove the following theorem:

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a real Hilbert space H . For $i = 1, \dots, m$, let $T_i : K \rightarrow CB(K)$ be a multi-valued k_i -strictly pseudo-contractive mapping. Let $\{x_n\}$ be the sequence defined by (3.1). If $F \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in F$, then $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, \dots, m$.*

Proof. Let $x^* \in \bigcap_{i=1}^m F(T_i)$. By Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\| \lambda_0(x_n - x^*) + \sum_{i=1}^m \lambda_i(u_i^n - x^*) \right\|^2 \\ &= \lambda_0 \|x_n - x^*\|^2 + \sum_{i=1}^m \lambda_i \|u_i^n - x^*\|^2 - \sum_{i=1}^m \lambda_0 \lambda_i \|u_i^n - x_n\|^2 \\ &\quad - \sum_{1 \leq i < j}^m \lambda_i \lambda_j \|u_i^n - u_j^n\|^2. \end{aligned}$$

Using the fact that for all $i = 1, \dots, m$, $T_i p = \{p\}$ for all $p \in F$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \lambda_0 \|x_n - x^*\|^2 + \sum_{i=1}^m \lambda_i (D(T_i x_n, T_i x^*))^2 - \sum_{i=1}^m \lambda_0 \lambda_i \|u_i^n - x_n\|^2 \\ &\quad - \sum_{1 \leq i < j}^m \lambda_i \lambda_j \|u_i^n - u_j^n\|^2. \end{aligned}$$

Since for each $i = 1, \dots, m$ T_i is k_i -strictly pseudo contractive, we obtain

$$\|x_{n+1} - x^*\|^2 \leq \lambda_0 \|x_n - x^*\|^2 + \sum_{i=1}^m \lambda_i (\|x_n - x^*\|^2 + k_i \|u_i^n - x_n\|^2)$$

$$\begin{aligned}
& - \sum_{i=1}^m \lambda_0 \lambda_i \|u_i^n - x_n\|^2 - \sum_{1 \leq i < j}^m \lambda_i \lambda_j \|u_i^n - u_j^n\|^2 \\
& \leq \|x_n - x^*\|^2 - \sum_{i=1}^m \lambda_i (\lambda_0 - k_i) \|u_i^n - x_n\|^2. \tag{3.3}
\end{aligned}$$

So,

$$\sum_{i=1}^m \lambda_i (\lambda_0 - k_i) \|u_i^n - x_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

This implies that

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^m \lambda_i (\lambda_0 - k_i) \|u_i^n - x_n\|^2 \right) < \infty.$$

Therefore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^m \lambda_i (\lambda_0 - k_i) \|u_i^n - x_n\|^2 = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \|x_n - u_i^n\| = 0, \quad \forall i = 1, \dots, m.$$

Since $u_i^n \in T_i x_n$, then

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0 \text{ for all } i = 1, \dots, m.$$

This completes the proof. \square

For $m = 1$, we get the result of Chidume et al. [26].

Corollary 3.3. *Let K be a nonempty, closed and convex subset of a real Hilbert space H . Suppose that $T : K \rightarrow CB(K)$ is a multi-valued k -strictly pseudo-contractive mapping. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. If $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Then, $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

We have the following corollary for finite family of multivalued nonexpansive mappings that improves and generalizes, in the case of Hilbert spaces, the results of Abbas et al. [33] and those of Rashwan and Altwqi [21].

Corollary 3.4. *Let K be a nonempty, closed and convex subset of a real Hilbert space H . Let $T_i, i = 1, \dots, m : K \rightarrow CB(K)$ be nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (3.1). If $F \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in F$, then $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, \dots, m$.*

We now approximate common fixed points of the mappings T_i through strong convergence of the sequence $\{x_n\}$ defined by (3.1). We start with the following definition:

Definition 3.5. A mapping $T : K \rightarrow CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p \in K$. We note that if K is compact, then every multi-valued mapping $T : K \rightarrow CB(K)$ is hemicompact.

Theorem 3.6. *Let K be a nonempty, closed and convex subset of a real Hilbert space H . For $i = 1, \dots, m$, let $T_i : K \rightarrow CB(K)$ be a multi-valued continuous k_i -strictly pseudo-contractive mapping. Assume that T_{i_0} is hemicompact for some i_0 . If $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of the T_i 's.*

Proof. From Theorem 3.2, we have that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$. Since T_{i_0} is hemicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p$ for some $p \in K$. Since T_i is continuous, we have $d(x_{n_j}, T_i x_{n_j}) \rightarrow d(p, T_i p)$. Therefore, $d(p, T_i p) = 0$ and so $p \in F(T_i)$. Setting $x^* = p$ in the proof of Theorem 3.2, it follows from inequality (3.3) that the sequence $\{\|x_n - p\|\}$ is decreasing and bounded from below. Therefore, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p . This completes the proof. □

Corollary 3.7. *Let K be a nonempty, compact and convex subset of a real Hilbert space H . For $i = 1, \dots, m$, let $T_i : K \rightarrow CB(K)$ be a multi-valued k_i -strictly pseudo-contractive mapping. Assume that for each i , T_i is continuous. If $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of the T_i 's.*

Proof. Observing that if K is compact, every map $T_i : K \rightarrow CB(K)$, $i = 1, \dots, m$ is hemicompact, the proof follows from Theorem 3.6. □

Definition 3.8. The mappings $T_1, \dots, T_m : K \rightarrow CB(K)$ satisfy condition (I^*) , if there exists a strictly increasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and $i_0, 1 \leq i_0 \leq m$ such that

$$d(x, T_{i_0} x) \geq f(d(x, F)), \quad \forall x \in K.$$

Theorem 3.9. *Let K be a nonempty, closed and convex subset of a real Hilbert space H . For $i = 1, \dots, m$, let $T_i : K \rightarrow CB(K)$ be a multi-valued k_i -strictly pseudo-contractive mapping. Assume that T_1, \dots, T_m satisfy condition (I^*) . If $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the T_i 's.*

Proof. From Theorem 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$. Using the fact that T_i satisfies condition (I) , it follows that $\lim_{n \rightarrow \infty} f(d(x_n, F(T_i))) = 0$. Thus, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T_i)$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j}, \quad \forall j \in \mathbb{N}.$$

By setting $x^* = p_j$ and following the same arguments as in the proof of Theorem 3.2, we obtain from inequality (3.3) that

$$\|x_{n_{j+1}} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in K . Notice that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}. \end{aligned}$$

This shows that $\{p_j\}$ is a Cauchy sequence in K and thus converges strongly to some $p \in K$. Using the fact that T_i is L -Lipschitzian and $p_j \rightarrow p$, we have

$$\begin{aligned} d(p_j, T_i p) &\leq D(T_i p_j, T_i p) \\ &\leq L \|p_j - p\|, \end{aligned}$$

so that $d(p, T_i p) = 0$ and thus $p \in T_i p$. Therefore, $p \in F(T_i)$ and $\{x_{n_j}\}$ converges strongly to p . Setting $x^* = p$ in the proof of Theorem 3.2, it follows from inequality (3.3) that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p . This completes the proof. \square

4. Applications

Let H be a real Hilbert space H and $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\varphi : H \rightarrow 2^H$ be a multivalued mapping. Consider the following optimization problem:

$$(P) \quad \begin{cases} \min f(x) \\ 0 \in \varphi(x). \end{cases}$$

Let us introduce the following multivalued maps $T_1, T_2 : H \rightarrow 2^H$ defined by

$$T_1 : I - \partial f \text{ and } T_2 := I - \varphi.$$

Theorem 4.1. *Let H be a real Hilbert space H and $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\varphi : H \rightarrow 2^H$ be a multivalued mapping. Assume that: (i) T_1 is k_1 -strictly pseudo-contractive and T_2 is k_2 -strictly pseudo-contractive; (ii) $F(T_1) \cap F(T_2) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in F(T_1) \cap F(T_2)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in H$ by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \lambda_0 x_n + \lambda_1 u_n + \lambda_2 v_n, \end{cases} \quad (4.1)$$

where $u_n \in T_1 x_n, v_n \in T_2 x_n$ and $\lambda_0 \in (k, 1), \lambda_1, \lambda_2 \in (0, 1)$ with $k := \max(k_1, k_2)$ and such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$. Then,

$$\lim_{n \rightarrow \infty} d(x_n, T_1 x_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_n, T_2 x_n) = 0.$$

Remark 2. The recursion formula (3.1) used in our theorems is of the Krasnoselkii type (see, e.g., [36]) and it is superior to the recursion formula of Abbas et al. (1.7) and the one of Rashwan and Altwqin [21] the following sense:

- In our algorithm, $u_i^n \in T_i x_n$ for $i = 1, \dots, m$ and do not have to satisfy the restrictive conditions (1.8) in the recursion formula (1.7) and similar restrictions (1.10) in the recursion formulas (1.9).

- The recursion formula (3.1) requires less computation time than the formula (1.7) because the parameters λ_i in formula (3.1) are fixed in $(k, 1)$, whereas in the algorithms (1.7) and (1.9), the λ_i are replaced by sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ in $(0, 1)$ satisfying the condition: $a_n + b_n + c_n = 1$. The parameters a_n , b_n , and c_n must be chosen at each step of the iteration process.

- The Krasnoselskii-type algorithm usually yields rate of convergence as fast as that of a geometric progression, whereas the algorithm (1.7), usually has order of convergence of the form $o(1/n)$.

Remark 3. In the Hilbert spaces setting, our theorems in this papers are important generalizations of several important recent results in the following sense: Our theorems extend results proved for multi-valued *nonexpansive* mappings in *real Hilbert spaces* (see, e.g., [28, 29, 31, 32, 33]) to the much more larger class of multi-valued *strictly pseudo-contractive mappings*.

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